# An Inhomogeneous Khintchine-Groshev Theorem

without monotonicity

joint work with Felipe Ramirez (Wesleyan)

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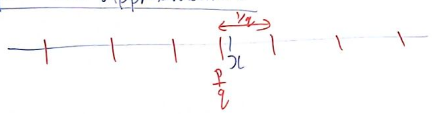
## Diophantine Approximation

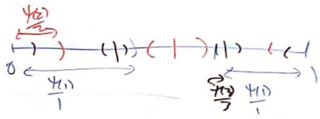



**Theorem** (Dirichlet, 1842)

For any $x \in \mathbb{R}$, $\exists$ i.m. $q \in \mathbb{N}$ s.t.

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

for some $p \in \mathbb{Z}$.




Given $\psi : \mathbb{N} \to [0, \infty)$, let

$$A(\psi) = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Khintchine's Theorem (1924) For any $\psi : \mathbb{N} \to [0,\infty)$,

$$\mathcal{L}(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ \boxed{1 \quad \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi \text{ is monotonic.}} \end{cases}$$

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First Borel-Cantelli Lemma Let $(X, \mu)$ is a measure space and suppose $(A_q)_{q=1}^{\infty}$ is a sequence of measurable subsets of $X$. If $\sum \mu(A_q) < \infty$, then $\mu\left(\limsup_{q\to\infty} A_q\right) = 0$.

$$\bigcap_{N=0}^{\infty} \bigcup_{q \geq N} A_q = \{x \in X : x \in A_q \text{ for i.m. } q \in \mathbb{N}\}$$

Proof of convergence part of KT

For $q \in \mathbb{N}$, let

$$A_q = \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1].$$

Then $\mathcal{A}(\psi) = \limsup_{q\to\infty} A_q$.

Note that

$$\mu(A_q) \leq q \times \frac{2\psi(q)}{q} = 2\psi(q)$$

$\underline{Q}$ : Do we really need monotonicity in Khintchine's Theorem?

$\underline{A}$ : YES. (Duffin-Schaeffer, 1941).

Let

$$A'(\psi) = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \text{ with } \gcd(p,q) = 1 \right\}.$$

<u>Duffin-Schaeffer Conjecture (1941)</u> For $\psi : \mathbb{N} \to [0,\infty)$,

$$\mathcal{L}(A'(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} < \infty, \\ 1 & \text{if } \quad " \quad = \infty. \end{cases}$$

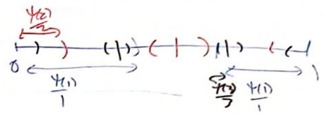
where $\varphi(n) = \#\{1 \le k \le n : \gcd(k,n) = 1\}$

Theorem (Koukoulopoulos + $\boxed{\text{Maynard}}$ 2020)

The DSC is true.

Gallagher (1961): $\mathcal{L}(A'(\psi)) = 0$ <u>or</u> 1.

Cassels (1950): $\mathcal{L}(A(\psi)) = 0$ <u>or</u> 1.




Given $\psi : \mathbb{N} \to [0,\infty)$, Let

$$A(\psi) = \left\{ x \in [0,1] : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

# Khintchine – Groshev Theorem

- Let $n, m \geq 1$ be integers.
- Let $\psi : \mathbb{N} \to [0, \infty)$.
- Let $\|\cdot\|$ denote the sup. norm.
- Define

$$A_{n,m}(\psi) = \Big\{ \overset{n\times m \text{ matrix}}{X} \in [0,1]^{nm} : \| \overset{n\text{-dim. row}}{\underline{q}} X - \underset{m\text{-dim. row}}{\underline{p}} \| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p}, \underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \Big\}.$$

<u>Khintchine – Groshev Theorem (1938)</u> For $\psi : \mathbb{N} \to [0, \infty)$,

$$\mathcal{L}(A_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

<u>Q</u>: Do we really need monotonicity here?

<u>A</u>: YES when $n = m = 1$, NO otherwise

☆ Duffin–Schaeffer (1941): $n = m = 1$.

☆ Gallagher (1965): $n = 1, m \geq 2$.

☆ Sprindžuk (1979): $n \geq 3$

☆ Beresnevich + Velani (2010): $nm > 1$.

## Inhomogeneous Khintchine – Groshev Theorem

- Let $n, m \geq 1$ be integers.
- Let $\psi : \mathbb{N} \to [0, \infty)$.
- Let $y \in \mathbb{R}^m$.
- Define

Szüsz (1958)
Schmidt (1964)
Sprindžuk (1979)

$n \times m$ matrix; $n$-dim. row; $m$-dim. row

$$A^{y}_{n,m}(\psi) = \{ \underline{x} \in [0,1]^{nm} : \| \underline{q}\underline{x} - \underline{p} - \underline{y} \| < \psi(\|\underline{q}\|) \text{ for i.m. } (\underline{p}, \underline{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n \}.$$

**Inhomogeneous Khintchine – Groshev Theorem** [Szüsz (1958), Schmidt (1964), Sprindžuk (1979)] For $\psi : \mathbb{N} \to [0, \infty)$, and any $y \in \mathbb{R}^m$,

$$\mathcal{L}(A^{y}_{n,m}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Q: Do we really need monotonicity here?

- ☆ Duffin-Schaeffer (1941): YES when $n = m = 1$. Ramírez (2017).
- ☆ Sprindžuk (1979): NO when $n \geq 3$.
- ☆ Yu (2021): NO when $n = 1, m \geq 3$.
- ☆ A.-Ramírez (2022): NO when $nm > 2$.